

# ON THE ENERGY EQUALITY FOR WEAK SOLUTIONS OF THE 3D NAVIER-STOKES EQUATIONS

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**ABSTRACT.** We prove that the energy equality holds for weak solutions of the 3D Navier-Stokes equations in the functional class  $L^3([0, T); V^{5/6})$ , where  $V^{5/6}$  is the domain of the fractional power of the Stokes operator  $A^{5/12}$ .

## 1. INTRODUCTION

We consider the 3D incompressible Navier-Stokes equations (NSE)

$$(1) \quad \begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \\ \nabla \cdot u = 0 \\ u(x, t) = 0 \text{ for } x \in \partial\Omega, \end{cases}$$

on an open bounded domain  $\Omega \subset \mathbf{R}^3$  of class  $C^2$ . Here  $u(x, t)$ , the velocity, and  $p(x, t)$ , the pressure, are unknowns;  $f(x, t)$  is a given driving force which we assume to belong to  $L^1([0, T); L^2(\Omega))$ , and  $\nu > 0$  is the kinematic viscosity coefficient of the fluid.

By the classical result of Leray [7] and Hopf [4], for every divergence-free initial data  $u_0 \in L^2(\Omega)$  and  $T > 0$  there exists a weak solution to the system (1) in the class

$$(2) \quad \mathcal{LH} = L_{\text{loc}}^\infty([0, T); L^2(\Omega)) \cap L_{\text{loc}}^2([0, T); H^1(\Omega)),$$

satisfying

$$(3) \quad \begin{aligned} \int_0^T \{-(u, \partial_t \varphi) + \nu(\nabla u, \nabla \varphi) + (u \cdot \nabla u, \varphi)\} dt \\ = (u_0, \varphi(0)) + \int_0^T (f, \varphi) dt, \end{aligned}$$

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for all test functions  $\varphi \in C_0^\infty([0, T) \times \Omega)$  with  $\nabla \cdot \varphi = 0$ . In addition one can find a weak solution satisfying the strong energy inequality, i.e.

$$(4) \quad |u(t)|_2^2 + 2\nu \int_{t_0}^t |\nabla u(s)|_2^2 ds \leq |u(t_0)|_2^2 + \int_{t_0}^t (f(s) \cdot u(s)) ds,$$

for all  $t \in [0, T)$  and almost all  $t_0 \in [0, t)$  including  $t_0 = 0$ . With an additional correction of  $u$  on a subset of  $[0, T)$  of measure zero one can ensure that  $u$  is weakly continuous with values in  $L^2(\Omega)$  (see Serrin [9]), and (3) holds in a stronger form,

$$(5) \quad (u(t), \varphi(t)) + \int_0^t \{-(u, \partial_s \varphi) + \nu(\nabla u, \nabla \varphi) + (u \cdot \nabla u, \varphi)\} ds \\ = (u_0, \varphi(0)) + \int_0^t (f, \varphi) ds,$$

for all  $t \in [0, T)$  and  $\varphi$  as before.

Solutions satisfying all the properties listed above are commonly called Leray-Hopf solutions. It is a famous open problem to show that a given Leray-Hopf solution to the Navier-Stokes system actually satisfies the energy equality

$$(6) \quad |u(t)|_2^2 + 2\nu \int_{t_0}^t |\nabla u(s)|_2^2 ds = |u(t_0)|_2^2 + \int_{t_0}^t (f(s) \cdot u(s)) ds,$$

for all  $0 \leq t_0 \leq t < T$  (or equivalently all  $t$  and  $t_0 = 0$ ). Continuing interest to this problem is motivated by the fact that the failure of (6) opens a possibility for an energy sink other than the natural viscous dissipation. Such a property of real fluids is not expected to exist physically. It has therefore been the subject of many works in the past and in recent years to find sufficient conditions for the energy equality to hold in bounded and unbounded domains. Below we mention the results that are most relevant to ours.

Serrin [9] showed (6) under the assumption that

$$(7) \quad u \in \mathcal{LH} \cap L^r([0, T); L^s), \text{ for } s \geq 3, \frac{3}{s} + \frac{2}{r} = 1.$$

Lions and Ladyzhenskaya improved the scaling in Serrin's condition to  $u \in L^4([0, T); L^4)$  (see [6, 8]). Shinbrot in [10] proved (6) under the extrapolated version of the Lions-Ladyzhenskaya condition, namely  $2/s + 2/r \leq 1$  for  $s \geq 4$ . One can show that Shinbrot's condition in the functional class  $\mathcal{LH}$  is weaker than that of Serrin (see [3]). A dimensionally the same, yet formally weaker, condition was recently found by Kukavica [5]. He shows that (6) holds if the pressure  $p$  is

locally  $L^2$ -summable in space-time. Since  $p$  is given by a Calderon-Zygmund operator applied to  $u \otimes u$ , Kukavica's condition is implied by that of Lions and Ladyzhenskaya.

In this present paper we prove (6) in a functional class with a better scaling than that of Shinbrot. Let us denote

$$(8) \quad H = \{u \in L^2(\Omega) : \nabla \cdot u = 0, u \cdot n|_{\partial\Omega} = 0\},$$

and let  $\mathbb{P} : L^2(\Omega) \rightarrow H$  be the  $L^2$ -orthogonal projection, referred to as the Helmholtz-Leray projector. Let  $A$  be the Stokes operator defined by

$$(9) \quad Au = -\mathbb{P}\Delta u.$$

The Stokes operator is a self-adjoint positive sectorial operator with a compact inverse. We denote  $V^s = \mathcal{D}(A^{s/2})$ ,  $s > 0$ , the domain of the fractional power of  $A$  (see the next section for details). Roughly,  $V^s$  corresponds to the Sobolev space  $H^s$ , and is in fact equal to  $H^s$  under periodic boundary conditions. In particular one can give the following description of  $V = V^1$  (see [2]):

$$(10) \quad V = \{u \in H^1(\Omega) : \nabla \cdot u = 0, u|_{\partial\Omega} = 0\}.$$

We prove the following result.

**Theorem 1.1.** *Every weak solution  $u(t)$  of (1) satisfying (5) on  $[0, T]$  with  $u \in \mathcal{LH} \cap L^3([0, T]; V^{5/6})$  verifies the energy equality (6).*

The dimensional  $L^r L^s$ -analogue of the functional class  $L^3([0, T]; V^{5/6})$  is  $L^3 L^{9/2}$ , which exhibits the scaling

$$(11) \quad 2/s + 2/r = 10/9.$$

It would be an interesting next step to try to prove the energy equality under this new scaling in the  $L^r L^s$ -class itself.

Finally we note that under periodic boundary conditions the result of Theorem 1.1 follows from – although not explicitly stated in – the recent work by the authors on Onsager's conjecture for the Euler equations [1]. So, the novelty of this present paper consists in the extension of Theorem 1.1 to the case of the Dirichlet boundary condition in a smooth domain.

## 2. PRELIMINARIES

First, let us introduce some notations and functional setting. Denote by  $(\cdot, \cdot)$  and  $|\cdot|$  the  $L^2(\Omega)$ -inner product and the corresponding  $L^2(\Omega)$ -norm. Let  $H$  and  $V$  be as above (8), (10), and  $V'$  denote the dual of  $V$ . We endow  $V$  with the norm  $\|u\| = |\nabla u|$ . There exists an orthonormal

basis of eigenvectors  $\{w_n\}$  in  $H$ , and a sequence of positive eigenvalues  $\{\lambda_n\}$ , such that

$$(12) \quad Aw_n = \lambda_n w_n, \quad w_n \in \mathcal{D}(A),$$

and

$$(13) \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty.$$

Any  $u \in H$  can then be written as

$$(14) \quad u = \sum_{n=1}^{\infty} (u, w_n) w_n.$$

Henceforth we will use the notation  $u_n = (u, w_n)$ . For  $s > 0$  we define the operator  $A^s$  by

$$(15) \quad A^s u = \sum_{n=1}^{\infty} \lambda_n^s u_n w_n,$$

and the space

$$(16) \quad V^s = \{u \in H : u = \sum_{n=1}^{\infty} u_n w_n, \|u\|_s^2 = \sum_{n=1}^{\infty} \lambda_n^s |u_n|^2 < \infty\}.$$

Thus,  $V^s = \mathcal{D}(A^{s/2})$ .

Now denote  $B(u, v) := \mathbb{P}(u \cdot \nabla v) \in V'$  for  $u, v \in V$ . So, we can rewrite (1) as the following differential equation in  $V'$ :

$$(17) \quad \partial_t u + \nu A u + B(u, u) = g,$$

where  $u$  is a  $V$ -valued function of time and  $g = \mathbb{P}f$ . Finally, we denote  $b(u, v, w) = \langle B(u, v), w \rangle$ . This trilinear form is anti-symmetric:

$$b(u, v, w) = -b(u, w, v), \quad u, v, w \in V,$$

in particular,  $b(u, v, v) = 0$  for all  $u, v \in V$ .

### 3. THE PROOF OF THEOREM 1.1

Define

$$(18) \quad P_{\kappa} u = \sum_{n: \lambda_n \leq \kappa} u_n w_n, \quad u \in H.$$

Let  $u \in V^{\beta}$  and denote  $u^l = P_{\kappa} u$ ,  $u^h = u - u^l$ . Observe the following inequalities:

$$(19) \quad \begin{aligned} \|u^l\|_{\beta} &\leq \kappa^{\beta-\alpha} \|u^l\|_{\alpha} \\ \|u^h\|_{\alpha} &\leq \kappa^{\alpha-\beta} \|u^h\|_{\beta}, \end{aligned}$$

whenever  $\beta > \alpha$ .

**Lemma 3.1.** *Let  $u(t)$  be a weak solution of (1) on  $[0, T]$ . Then*

$$(20) \quad |u(t)|^2 + 2\nu \int_{t_0}^t \|u\|^2 ds = |u(t_0)|^2 + 2 \int_{t_0}^t (g, u) ds + 2 \lim_{\kappa \rightarrow \infty} \int_{t_0}^t b(u, u^l, u) ds,$$

for all  $0 \leq t_0 \leq t < T$ .

*Proof.* One can see directly from (5) that  $u^l \in C_w([0, T]; V)$  and  $\partial_t u^l \in L^2_{loc}([0, T]; V)$ . Thus, using  $u^l$  as a test function in (5) (allowed by a standard approximation argument) we obtain

$$(21) \quad |u^l(t)|^2 - |u^l(t_0)|^2 + 2\nu \int_{t_0}^t \|u^l\|^2 ds - 2 \int_{t_0}^t (g, u^l) ds = 2 \int_{t_0}^t b(u, u^l, u) ds.$$

From this we see that the limit of the right hand side exists as  $\kappa \rightarrow \infty$ , which completes the proof of the lemma.  $\square$

Let  $u^l$  and  $u^h$  be defined as before. In view of Lemma 3.1, it suffices to show that

$$(22) \quad \lim_{\kappa \rightarrow \infty} \int_0^T |b(u, u^l, u)| ds = 0.$$

Indeed, let us write

$$b(u, u^l, u) = b(u^h, u^l, u^h) + b(u^l, u^l, u^h) + b(u^h, u^l, u^l) + b(u^l, u^l, u^l).$$

The last two terms vanish, so it suffices to estimate only the first two. We use the standard estimate found, for example, in [2]:

$$(23) \quad |b(u, v, w)| \leq \|u\|_{s_1} \|v\|_{s_2+1} \|w\|_{s_3}$$

where  $s_1 + s_2 + s_3 \geq 3/2$ . To estimate the first term let us set  $s_1 = s_2 = s_3 = 1/2$ , then

$$|b(u^h, u^l, u^h)| \leq \|u^h\|_{1/2}^2 \|u^l\|_{3/2},$$

and by (19) we have

$$(24) \quad \|u^h\|_{1/2} \leq \kappa^{-1/3} \|u^h\|_{5/6}$$

$$(25) \quad \|u^l\|_{3/2} \leq \kappa^{2/3} \|u^l\|_{5/6}.$$

So,

$$|b(u^h, u^l, u^h)| \leq \|u^h\|_{5/6}^2 \|u^l\|_{5/6},$$

which tends to zero a.e. in  $t$  as  $\kappa \rightarrow \infty$ . Since in addition,

$$|b(u^h, u^l, u^h)| \leq \|u\|_{5/6}^3$$

for all  $t$ , by the Dominated Convergence Theorem,

$$|b(u^h, u^l, u^h)| \rightarrow 0, \quad \text{as } \kappa \rightarrow \infty,$$

in  $L^1([0, T])$ . As to the second term, similar estimates with  $s_1 = 5/6$ ,  $s_2 = 0$ ,  $s_3 = 2/3$ , yield

$$|b(u^l, u^l, u^h)| \leq \|u^l\|_{5/6}^2 \|u^h\|_{5/6},$$

which also tends to zero in  $L^1([0, T])$  as  $\kappa \rightarrow \infty$ . Hence we obtain the desired energy equality.

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